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A note on f^+-chromatic spanning trees in f-properly edge-colored graphs

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Abstract

A rainbow tree is an edge-colored tree whose all edges are colored with different colors. An f-chromatic tree is an edge-colored tree such that each color c appears on at most $f(c)$ edges. A properly edge-colored graph is an edge-colored graph whose any adjacent edges are colored with different colors. An f-properly edge-colored graph is an edge-colored graph such that each vertex is incident with at most $f(c)$ edges colored with c for any color c.

In this paper, we prove that every f-properly edge-colored graph G has k edge-disjoint f^+-chromatic spanning trees under the assumption that for any proper edge-coloring of G, there exist k edge-disjoint rainbow spanning trees in the properly edge-colored graph G. Here f^+ is a mapping such that $f^+(c) = f(c)+1$ for any color c. By using this theorem, we show that every f-properly edge-colored complete graph of order n ($n \geq 5$) has two edge-disjoint f^+-chromatic spanning trees.

We consider finite undirected graphs without loops or multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. For a graph G and a set $\mathcal{C}$ of colors, let *color* be a mapping from $E(G)$ to $\mathcal{C}$. Then, the mapping *color* is called an *edge-coloring* of G, and the triple $(G, \mathcal{C}, color)$ is called an *edge-colored graph*. We often abbreviate an edge-colored graph $(G, \mathcal{C}, color)$ as G. An edge-coloring of a graph is said to be *proper* if any adjacent edges are colored with different colors. An edge-colored graph is said to be *properly edge-colored* if its edge-coloring is proper. For a graph G, $\Delta(G)$ denotes the maximum degree of G, and $\chi'(G)$ denotes the edge chromatic number (chromatic index) of G, namely, the minimum positive integer k such that there exists a proper edge-coloring of G using

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k colors. Note that $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$ by Vizing's theorem [4]. For an edge-colored graph $(G, \mathcal{C}, \mathit{color})$, let $E_c(G)$ denote the set of all the edges of G colored with $c \in \mathcal{C}$, and let $G_c := (V(G), E_c(G))$. An edge-colored graph $(G, \mathcal{C}, \mathit{color})$ is said to be *rainbow*[1] if no two edges of G have the same color, that is, $\mathit{color}(e) \neq \mathit{color}(e')$ for any two distinct edges e and e' of G.

Brualdi and Hollingsworth [1] conjectured the existence of n edge-disjoint rainbow spanning trees in some type of properly edge-colored complete graphs (Fig. 1).

Conjecture 1 (Brualdi and Hollingsworth [1]). *Let $(G, \mathcal{C}, \mathit{color})$ be an edge-colored complete graph of order $2n$ ($n \geq 3$). If each color induces a 1-factor of G, that is, $E_c(G)$ is a perfect matching of G for any color $c \in \mathcal{C}$, then G can be decomposed into n edge-disjoint rainbow spanning trees.*

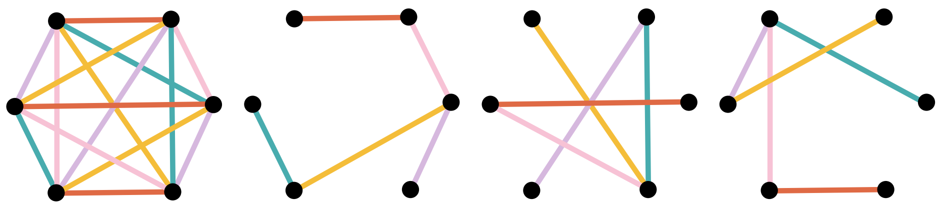

Fig. 1: An example for $n = 3$ of the conjecture by Brualdi and Hollingsworth.

Kaneko et al. [2] conjectured a stronger statement than Conjecture 1 and proved the existence of two edge-disjoint rainbow spanning trees.

Conjecture 2 (Kaneko, Kano, and Suzuki [2]). *Every properly edge-colored complete graph of order n ($n \geq 5$) has $\lfloor n/2 \rfloor$ edge-disjoint rainbow spanning trees.*

Theorem 3 (Kaneko, Kano, and Suzuki [2]). *Every properly edge-colored complete graph of order n ($n \geq 5$) has two edge-disjoint rainbow spanning trees.*

The term "rainbow" means that each color appears on at most one edge. Suzuki [3] generalized "one" to a mapping f from a given color set $\mathcal{C}$ to the set of non-negative integers, and defined f*-chromatic* graphs as follows.

Definition 4 (Suzuki [3]). *Let $(G, \mathcal{C}, \mathit{color})$ be an edge-colored graph. Let f be a mapping from $\mathcal{C}$ to the set of non-negative integers. Then $(G, \mathcal{C}, \mathit{color})$ is said to be f-chromatic if $|E_c(G)| \leq f(c)$ for any color $c \in \mathcal{C}$.*

Suzuki [3] presented a necessary and sufficient condition for an edge-colored graph to have an f-chromatic spanning forest with exactly m components.

[1] A rainbow graph is also said to be *heterochromatic*, *multicolored*, *totally multicolored*, *polychromatic*, or *colorful*, and so on.

In this paper, first, we prove the following theorem, which shows a relationship between the existence of rainbow spanning trees and the existence of some type of f-chromatic spanning trees.

Theorem 5. *Let $(G, \mathcal{C}, \mathit{color})$ be an edge-colored graph. Let f be a mapping defined as $f(c) := \chi'(G_c)$ for each color $c \in \mathcal{C}$. Suppose that for any proper edge-coloring color' with color set $\mathcal{C}'$ of G, the properly edge-colored graph $(G, \mathcal{C}', \mathit{color}')$ has k edge-disjoint rainbow spanning trees. Then, $(G, \mathcal{C}, \mathit{color})$ has k edge-disjoint f-chromatic spanning trees.*

Proof. The outline of our proof is as follows. We recolor all the edges of G in such a way that for each color $c \in \mathcal{C}$, we properly color the edges of G_c with $\chi'(G_c)$ new colors. Then, the recolored graph is properly edge-colored. By our assumption, the recolored graph has k edge-disjoint rainbow spanning trees. By restoring the colors of the trees, we have k edge-disjoint f-chromatic spanning trees of $(G, \mathcal{C}, \mathit{color})$.

More strictly, for each color $c \in \mathcal{C}$, let $(c, 1), (c, 2), \ldots, (c, \chi'(G_c))$ be new $\chi'(G_c)$ colors, let $\mathcal{C}_c$ be the set of these colors, and let color_c be a proper edge-coloring of G_c with colors in $\mathcal{C}_c$. Let $\mathcal{C}' := \bigcup_{c \in \mathcal{C}} \mathcal{C}_c$ and $\mathit{color}' := \bigcup_{c \in \mathcal{C}} \mathit{color}_c$. Then, color' be the new edge-coloring with color set $\mathcal{C}'$ of G such that $\mathit{color}'(e) = \mathit{color}_{\mathit{color}(e)}(e)$ for each edge e of G. Since every coloring color_c is proper and $\bigcap_{c \in \mathcal{C}} \mathcal{C}_c = \emptyset$, the edge-colored graph $(G, \mathcal{C}', \mathit{color}')$ is a properly edge-colored.

By our assumption, $(G, \mathcal{C}', \mathit{color}')$ has k edge-disjoint rainbow spanning trees $(T_i, \mathcal{C}', \mathit{color}'_i)$ $(i = 1, 2, \ldots, k)$, where $\mathit{color}'_i = \{(e, \mathit{color}'(e)) \mid e \in E(T_i)\} \subseteq \mathit{color}'$. For each $(T_i, \mathcal{C}', \mathit{color}'_i)$, we let $\mathit{color}_i := \{(e, \mathit{color}(e)) \mid e \in E(T_i)\} \subseteq \mathit{color}$ and we consider the edge-colored tree $(T_i, \mathcal{C}, \mathit{color}_i)$. For any color $c \in \mathcal{C}$, each $(T_i, \mathcal{C}', \mathit{color}'_i)$ has at most $|\mathcal{C}_c| = \chi'(G_c)$ edges whose color is (c, j) for some $j \in \{1, 2, \ldots, \chi'(G_c)\}$, since $(T_i, \mathcal{C}', \mathit{color}'_i)$ is rainbow. By the definition of color', $\mathit{color}'(e) = (c, j)$ for some $j \in \{1, 2, \ldots, \chi'(G_c)\}$ if and only if $\mathit{color}(e) = c$. Thus, $(T_i, \mathcal{C}, \mathit{color}_i)$ has at most $\chi'(G_c) = f(c)$ edges whose color is c, which implies $(T_i, \mathcal{C}, \mathit{color}_i)$ is f-chromatic. Therefore, G has k edge-disjoint f-chromatic spanning trees. □

A properly edge-colored graph $(G, \mathcal{C}, \mathit{color})$ is an edge-colored graph such that $\Delta(G_c) \leq 1$ for any color $c \in \mathcal{C}$. To generalize this fact, we define an *f-properly edge-colored* graph as follows.

Definition 6. *Let f be a mapping from a color set $\mathcal{C}$ to the set of non-negative integers. An edge-colored graph $(G, \mathcal{C}, \mathit{color})$ is said to be f-properly edge-colored if $\Delta(G_c) \leq f(c)$ for any color $c \in \mathcal{C}$. Then, the edge-coloring color is called an f-proper edge-coloring of G.*

Given an edge-colored graph $(G, \mathcal{C}, \mathit{color})$ and a mapping f from $\mathcal{C}$ to the set of non-negative integers, we let $f^+(c) := f(c) + 1$ for each color $c \in \mathcal{C}$. We next prove the following theorem by using Theorem 5 and Vizing's theorem [4].

Theorem 7. *Let $(G, \mathcal{C}, color)$ be an f-properly edge-colored graph. Suppose that for any proper edge-coloring $color'$ with color set $\mathcal{C}'$ of G, the properly edge-colored graph $(G, \mathcal{C}', color')$ has k edge-disjoint rainbow spanning trees. Then, $(G, \mathcal{C}, color)$ has k edge-disjoint f^+-chromatic spanning trees.*

Proof. Let g be a mapping defined as $g(c) \coloneqq \chi'(G_c)$ for each color $c \in \mathcal{C}$. By Theorem 5, $(G, \mathcal{C}, color)$ has k edge-disjoint g-chromatic spanning trees $T_1, T_2, \ldots, T_k$. Since $(G, \mathcal{C}, color)$ is f-properly edge-colored, $\Delta(G_c) \leq f(c)$ for any color $c \in \mathcal{C}$. Thus, by Vizing's theorem, $g(c) = \chi'(G_c) \leq \Delta(G_c) + 1 \leq f(c) + 1 \leq f^+(c)$ for each color $c \in \mathcal{C}$. Thus, a g-chromatic spanning tree of $(G, \mathcal{C}, color)$ is an f^+-chromatic spanning tree of $(G, \mathcal{C}, color)$. Hence, $T_1, T_2, \ldots, T_k$ are desired trees. □

By using Conjecture 2, and Theorem 3, we have the following corollaries of Theorem 7.

Corollary 8. *If Conjecture 2 holds, then every f-properly edge-colored complete graph of order n $(n \geq 5)$ has $\lfloor n/2 \rfloor$ edge-disjoint f^+-chromatic spanning trees.*

Corollary 9. *Every f-properly edge-colored complete graph of order n $(n \geq 5)$ has two edge-disjoint f^+-chromatic spanning trees.*

We conclude this note with the following stronger conjecture, which claims the existence of f-chromatic spanning trees instead of f^+-chromatic spanning trees.

Conjecture 10. *Every f-properly edge-colored complete graph of order n $(n \geq 5)$ has $\lfloor n/2 \rfloor$ edge-disjoint f-chromatic spanning trees.*

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